MTH 166

Lecture-22

Solution of Laplace Equation

Topic:

Solution of Partial Differential Equations

Learning Outcomes:

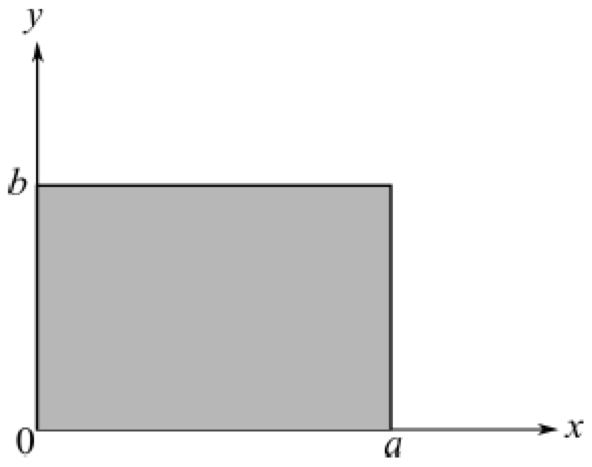
To solve two dimensional Laplace Equation

Problem. **Solve $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ [2D-Laplace Equation]**

Solution. The given one dimensional wave equation is:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad (1) \qquad 0\le x\le a, 0\le y\le b, t>0$$

Let solution be: $u(x,y)=XY$ (2) where $X=f(x), Y=g(y)$



$$\Rightarrow \frac{\partial^2 u}{\partial x^2}=X''Y \text{ and } \frac{\partial^2 u}{\partial y^2}=XY''$$

Equation (1) becomes: $X''Y+XY''=0$

$$\Rightarrow \frac{X''}{X}=-\frac{Y''}{Y}=k \quad \text{(Say)}$$

As k can take three values: zero, positive or negative, so we have following three cases.

<u>Case 1.</u> When $k = 0$

$$\frac{X''}{X} = k \qquad \Rightarrow \frac{X''}{X} = 0 \qquad \Rightarrow X'' = 0 \qquad \Rightarrow X = ax + b$$

Also $-\frac{Y''}{Y} = k \qquad \Rightarrow -\frac{Y''}{Y} = 0 \qquad \Rightarrow Y'' = 0 \qquad \Rightarrow Y = cy + d$

Required solution of equation (1) is:

$u(x, y) = XY = (ax + b)(cy + d)$

<u>Case 2.</u> When $k = p^2$ (Positive)

$$\frac{X''}{X} = k \qquad \Rightarrow \frac{X''}{X} = p^2 \qquad \Rightarrow X'' - p^2 X = 0$$

<u>S.F.</u> $(D^2 - p^2)X = 0$

<u>A.E.</u> $(D^2 - p^2) = 0 \qquad \Rightarrow D = \pm p$

$\therefore X = ae^{px} + be^{-px}$

Also $\quad -\frac{Y''}{Y} = k \quad \Rightarrow -\frac{Y''}{Y} = p^2 \quad \Rightarrow Y'' + p^2 Y = 0$

<u>S.F.</u> $(D^2 + p^2)Y = 0$

<u>A.E.</u> $(D^2 + p^2) = 0 \quad \Rightarrow D = \pm ip$

$\therefore Y = c\cos py + d\sin py$

Required solution of equation (1) is:

$u(x, y) = XY = (ae^{px} + be^{-px})(c\cos py + d\sin py)$

<u>Case 3.</u> When $k = -p^2$ (Negative)

$\frac{X''}{X} = k \quad \Rightarrow \frac{X''}{X} = -p^2 \quad \Rightarrow X'' + p^2 X = 0$

S.F. $(D^2 + p^2)X = 0$

A.E. $(D^2 + p^2) = 0 \qquad \Rightarrow D = \pm ip$

$\therefore X = a\cos px + b\sin px$

Also $-\frac{Y''}{Y} = k \qquad \Rightarrow -\frac{Y''}{Y} = -p^2 \qquad \Rightarrow Y'' - p^2 Y = 0$

S.F. $(D^2 - p^2)Y = 0 \ = 0$

A.E. $(D^2 - p^2) \ = 0 = 0 \qquad \Rightarrow D = \pm p$

$\therefore Y = ae^{py} + be^{-py}$

Required solution of equation (1) is:

$u(x, y) = XY = (a\cos px + b\sin px)\,(ae^{py} + be^{-py})$

Note: For Laplace Equation

Equation: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Nature: Elliptic

Solution: 1. $u(x,y) = (ax + b)(cy + d)$

2. $u(x,y) = (ae^{px} + be^{-px})(c\cos py + d\sin py)$

or

$$u(x,y) = (a\cosh px + b\sinh px)(c\cos py + d\sin py)$$

3. $u(x,y) = (a\cos px + b\sin px)\,(ae^{py} + be^{-py})$

or

$$u(x,y) = (a\cos px + b\sin px)\,(a\cosh py + b\sinh py)$$

Problem. Prove that a two dimensional heat equation becomes Laplace equation in steady state.

Solution. The two dimensional heat equation is:

$$\frac{\partial u}{\partial t} = C^2 \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] \qquad (1)$$

In steady-state: $\frac{\partial u}{\partial t} = 0$

So, in steady-state equation (1) becomes:

$$0 = C^2 \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] \quad \Rightarrow \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] = 0 \qquad (\text{As } C^2 \neq 0)$$

which is a two dimensional Laplace Equation.